Category Theory: How Can We Generalise Mathematics?

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Quick Poll

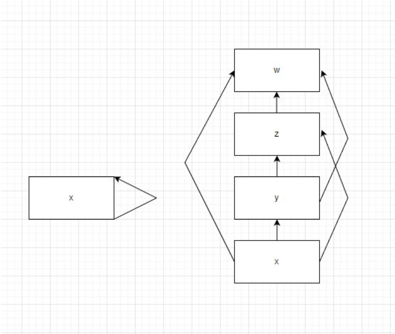
By a show of hands, who has taken 33A? 115A? 110AB? 210ABC?

Outline

Categories

What is a category? A collection of **objects** and **arrows** with three rules:

1. Every object has an identity arrow
2. Given two arrows that lie tail-to-head, their composition is an arrow
3. Composition of arrows is associative



Examples of Categories

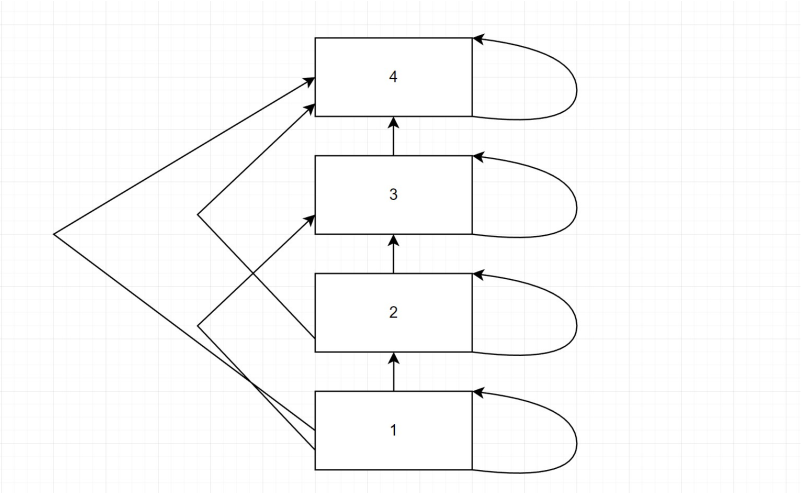
Examples:

1. Set (the category of sets and functions [1])
2. Grp (the category of groups and group homomorphisms)
3. $\mathbb{R}$ (the set of real numbers, with an arrow from x to y if $x \leq y$ – an example of a **poset** [partially-ordered set] category)

[1] there are in fact issues with having a set of all sets, but we are able to work around that in a way we will not discuss here – if you are interested, ask me after

Catergories are Just Graphs

Categories are just graphs. This is an example of $\mathbb{Z}^+$ as a poset category (obviously it should go off to infinity but I got bored of adding arrows).



Outline

Uses

You might be wondering, what's the point? Why do we care about putting normal things into weird graphs?
Algebra is the study of symmetries and category theory allows us to generalise those symmetries far more easily. Category theory is a tool that allows us to use information we know in a specific case and extend it to a vast number of related situations. It is a common language for all fields mathematics.

Examples

For the rest of this talk, we will take a look at two important uses of category theory across vastly different fields of mathematics: analysis and algebra.

Outline

Functors

Let's quickly define a very important categorical concept. Given two categories C and D, a function between them is called a **functor** $F : \mathsf{C} \to \mathsf{D}$ and works as follows: given an object X of C, $F(X)$ is an object of D. Given an arrow $f : X \to Y$ in C, $F(f) : F(X) \to F(Y)$ is an arrow in D. And finally, F respects composition: $F(f \circ g) = F(f) \circ F(g)$.

Derivatives

As you hopefully remember from your multivariable calculus or analysis class, given a function $f : \mathbb{R}^n \to \mathbb{R}^m$, we can find its "total derivative", Df, which is a linear map from $\mathbb{R}^n \to \mathbb{R}^m$. You may be surprised to find that D is a functor $D : \mathsf{Man}^\infty \to \mathsf{Man}^\infty$ (the category of smooth manifolds).

The exact idea behind manifolds is not important here, but remember that not every continuous function is differentiable, and we require Df to exist for any arrow in our category. So we choose a category where every arrow is differentiable. You can basically just treat Man^∞ as the category of spaces and differentiable functions between them.

Why Should I Care?

Because Df is a functor, it must preserve composition. Thus, $D(f \circ g) = (Df) \circ (Dg)$. If we say $f, g : \mathbb{R} \to \mathbb{R}$, then at any point $p \in \mathbb{R}$, we have $(D(f \circ g))(p) = (Df \circ Dg)(p) = f'(g(p)) \cdot g'(p)$. We've recovered the chain rule from a property of functors!

Outline

Algebraic Structures

Real quick, let's go over a couple algebraic structures so we can see how they interact in category theory. Let X be a set. We call X a **commutative monoid** if it has an operation $+$ satisfying the following conditions:

1. $x + y = y + x$
2. There is an identity 0 so that $x + 0 = 0$

We call X an **abelian group** if it is a commutative monoid and for any $x \in X$, there is $-x \in X$ so that $x + (-x) = 0$. We call X a **commutative ring** if it has an abelian group with respect to $+$ and it also has an operation $\times$ which makes it a commutative monoid. We call X a **field** if it is a commutative ring and it is also an abelian group with repsect to $\times$.

That was a lot of words. Let's look at examples.

1. $\mathbb{N}$ (the natural numbers) is a commutative monoid, with operation $+$ (addition) and identity 0
2. $\mathbb{Z}$ (the integers) is an abelian group with operation $+$
3. $\mathbb{Z}$ (the integers again) is a commutative ring with operations $+, \times$
4. $\mathbb{Q}$ (the rational numbers) is a field with operations $+, \times$

Adjunctions

Adjunctions are one of the most important concepts in category theory. Given two categories C and D, and functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$, we say F and G form an **adjunction** if

$$\mathrm{Hom}_{\mathsf{D}}(FX, Y) \simeq \mathrm{Hom}_{\mathsf{C}}(X, GY)$$

For any $X \in \mathsf{C}, Y \in \mathsf{D}$.

Examples of Adjunctions

Adjunctions are a powerful way to explain relations that were first discovered hundreds of years earlier. Free groups are the “simplest groups.” Free rings are the “simplest rings.” How can we explain and generalise that?

$$\mathsf{Grp} \underset{Free}{\overset{Forgetful}{\rightleftarrows}} \mathsf{Set} \quad (\perp)$$

$$\mathsf{Rng} \underset{Free}{\overset{Forgetful}{\rightleftarrows}} \mathsf{Set} \quad (\perp)$$

Beauty of Adjunctions

Adjunctions are really a beautiful idea because they let us massively generalise a specific interaction between functors. If you've taken 115a, you may remember the adjoint of a linear operator is defined by $\langle T\vec{v}, \vec{w}\rangle = \langle \vec{v}, T^*\vec{w}\rangle$. This is a massive generalisation of that concept, but many of the relations between adjoint linear operators still apply to adjoint functors.
Adjoint functors are just one example of the beauty that category theory can bring to mathematics. Categories are truly the common thread that unite all of mathematics, providing the "hidden reason" why similar things really are similar and why everything works in such an amazing and beautiful way.

Conclusion

Thank you!